On the mobility and efficiency of mechanical systems

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Abstract

The definition of a mobilized system and its efficiency are introduced. The existence of an optimal (maximally efficient) system is proved by an application of Young measures and compensated compactness.

1 Introduction

1.1 Motivation

Which one is the exception- a motor boat, a car, an airplane or a submarine? The immediate (perhaps, after a short contemplation) answer is, of course, the car. Indeed, a car is the only vehicle in the above list which "defies" the law of conservation of linear momentum. The others create an opposite stream in the medium in which they move (air, water) which, by conservation of linear momentum, must be compensated by the motion of the vehicle. Take away the friction of these vehicles and the medium, and they will perform better (faster, more efficient).

The principle behind the motion of a car is different. It is moving (that is, shifting from rest to cruise velocity) *because* of the friction created by the contact of its tires with the road. Take away the friction and the car will not be able to move at all, independently of how hard you push the gas pedal. You will not be able to stop either, if your friend, unwisely attempting to help, gave you an initial push on a frictionless road.

So, what is a car? If we strip it off the non-essential components (radio, GPS, the fashioned seat covers etc.), it is a collection of components which can move with respect to each other under the preassign constraints caused by the mechanical structure. Unless you change gear or the pressure you apply on the gas pedal, the motion of these inner part can be assumed to be periodic (again, with respect to the frame of reference of the car itself). This concept has a lot in common with the subject of optimal locomotion of a swimmer in a Stokes flow - where the motion is due to a periodic change of shape of the swimmer in the absence of inertia. The description below is, in a sense, the mechanical analog of the swimmer model, see [AGK] and the references listed there. Another series of publications which seems to be related to the present discussion concerns molecular motors and the flashing rachet. See [CHK], [CKK], DKK] and ref. therein.

Note that in the case of micro-swimmers and molecular motors, it seems that there is no intuitive way to predict the direction and velocity of the swimmer (res. motor) from its periodic motion. A car, at a first glance, is different. However, the abstraction of a car which we consider in this paper (and call a "mechanical system") is, in a way, a generalization of

the concept which contains cars, microswimers, molecular motors and, perhaps, many other objects whose dynamics are not inertial in nature.

1.2 Objectives and outline of results

A mechanical system is represented by a Lagrangian $L = L(\dot{x}, t)$. It is $T-$periodic in the time t, and $x(t)$ stands for the position of a reference point (say, the center of mass of the system). Such a system is called "mobilized" if the global minimizer $x(t)$ of the action $\int_0^T L(\dot{x}, t)dt$ is *not* periodic, i.e $x(T) \neq x(0)$. In section 2 we attempt to justify this model and explain the reason why the global minimum of the action represents the asymptotic motion of the system under friction. In addition, we introduce a reasonable definition of efficiency, in terms of a relation involving the speed $\overline{v} := |x(T) - x(0)|/T$ due to the action minimizer, and the minimal action $\overline{D} = \int_0^T L(\dot{x}, t)dt$ itself. The number $\overline{D}$ stands for some indication of the energy dissipated (or invested) per period, so a more efficient system means larger $\overline{v}$ and smaller $\overline{D}$. We scale the efficiency function e_L of a mechanical system L in such a way that $0 < e_L < 1$. Then, we ask if either there is a (theoretical) possibility to achieve the "most efficient" system $\overline{L}$ whose efficiency $\overline{e} := e_{\overline{L}} < 1$, or there is a sequence of systems L_n whose efficiencies $e_{L_n} \rightarrow \overline{e} \leq 1$, but the "ideal", most efficient system $\overline{L}$ does not exist.

In section 3 we consider a special case of mechanical systems, where L is a homogeneous function of $\dot{x}$, and prove the first alternative: There *is* a "best" (most efficient) mechanical system and its efficiency $\overline{e}$ is always smaller than 1.

2 Description of the model

Let us attempt to build a mathematical caricature the "car", composed of a finite number of "parts". To wit, assume it is composed of a collection of N points of respective masses m_i executing orbits $x_i = x_i(t)$ on the line $\mathbb{R}$, $i = 1, \ldots n$, so that

$$x_i(t + T) = x_i(t) + T\overline{v} \tag{2.1}$$

where $\overline{v}$ is the effective velocity of the car (with respect to a reference frame attached to the road) and T is the period of one cycle.

We assume $\sum_1^N m_i = 1$. The orbit $x_i(t)$ is given with respect to a *fixed* frame of reference (attached to the road). We may write it as

$$x_i(t) = y_i(t) + x(t)$$

where y_i is the orbit of the corresponding point with respect to a reference attached to the car and $x = x(t)$ is the orbit of the car as a whole with respect to a reference attached to the road. In this representation y_i is a periodic orbit, representing the motion of a part forced by, say, the internal combustion of the engine.

The simplest model for the motion of $x(t)$ is the linear forced system

$$\ddot{x} + \beta\dot{x} = F(t) \tag{2.2}$$

where $\beta > 0$ is the friction coefficient and F is the total forcing acted on the car by the motion of its inner parts. It is the sum of the forces $F := \sum_i^N f_i$, where f_i, the force applied by the i part, is *defined* in terms of y_i as:

$$f_i(t) = -m_i(\ddot{y}_i + \beta\dot{y}_i) \ . \tag{2.3}$$

Now, define

$$\mathcal{L}^{\{y\}}(\dot{x}, t) := \sum_{i=1}^N m_i L(\dot{y}_i(t) + \dot{x}) \ . \tag{2.4}$$

Then (2.2, 2.3) can be summarized as the Euler-Lagrange equation corresponding to the Lagrangian

$$\mathcal{L}_\beta^{\{y\}}(\dot{x}, t) := e^{\beta t}\mathcal{L}^{\{y\}}(\dot{x}, t) \tag{2.5}$$

where

$$L(s) := |s|^2/2 \ . \tag{2.6}$$

Indeed, the Euler-Lagrange equation associated with $\mathcal{L}_\beta^{\{y\}}$ under (2.6) is

$$0 = e^{\beta t}\left[\sum_1^N m_i(\ddot{x} + \ddot{y}_i) + \beta\sum_1^N m_i(\dot{x} + \dot{y}_i)\right]$$

which implies (2.2) via (2.3) and the condition $\sum m_i = 1$.

The energy dissipated per cycle for an orbit $y_1, \dots y_N, x$ is given by βD where

$$D := \sum_{i=1}^N m_i \int_0^T \frac{|\dot{y}_i + \dot{x}|^2}{2} dt = \int_0^T \mathcal{L}^{\{y\}}(\dot{x}, t) dt \ . \tag{2.7}$$

We now generalize (2.4-2.7) into:

Definition 2.1. *A mechanical system $\mathbf{L}_\beta$ is determined by a forced orbit composed of N periodic functions $\boldsymbol{y} = \{y_i(t), \dots y_N(t)\}$ in terms of the Lagrangian $\mathcal{L}_\beta^{\{y\}}$ as given in (2.5) and a convex function L generalizing (2.6).*

The moment associated with $\mathcal{L}^{\{y\}}$ is denoted by $p = \mathcal{L}_{\dot{x}}^{\{y\}}$. The Euler-Lagrange equation associated with (2.5) is

$$\dot{p} + \beta p = 0 \Longrightarrow p(t) \to 0$$

so the asymptotic motion of a mechanical system $\mathbf{L}_\beta$ is determined by the orbit $p(t) \equiv 0$.

To elaborate, let

$$\mathcal{H}^{\{y\}}(p, t) = \sup_\zeta \left[p \cdot \zeta - \mathcal{L}^{\{y\}}(\zeta, t)\right] \ . \tag{2.8}$$

be the Hamiltonian associated with the Lagrangian $\mathcal{L}^{\{y\}}$. The equation of motion corresponding to the non-dissipative dynamics is given by

$$\dot{x}(t) = \mathcal{H}_p^{\{y\}}(\lambda, t)$$

where λ stands for the constant momentum p. If a friction is applied, then $x = x(t)$ is determined by $p = 0$, namely

$$\dot{x}(t) = \mathcal{H}_p^{\{y\}}(0,t)$$

is the asymptotic motion of the system. It is a *global* minimizer of the action determined by the Lagrangian $\mathcal{L}^{\{y\}}$. Note that

$$\min_{x=x(t)} \int_0^T \mathcal{L}^{\{y\}}(\dot{x},t)dt = -\int_0^T \mathcal{H}^{\{y\}}(0,t)dt \ .$$

Let

$$\overline{v}(0) = \frac{1}{T}\int_0^T \mathcal{H}_p^{\{y\}}(0,t)dt \equiv \frac{x(T)-x(0)}{T} \ ,$$

where $x(t)$ is the global minimizer of the action.

Definition 2.2. *A mechanical system for which $\overline{v}(0) \neq 0$ is called a mobilized system.*

The first result is somewhat disappointing:

Theorem 1. *If L is a quadratic function (2.6), then the system is not mobilized .*

Proof.

$$\mathcal{L}^{\{y\}}(\dot{x},t) = \frac{1}{2}\sum_{i=1}^N m_i\left|\dot{y}_i(t)+\dot{x}\right|^2 \Longrightarrow \mathcal{L}_{\dot{x}}^{\{y\}} = \sum_{i=1}^N m_i\left(\dot{y}_i(t)+\dot{x}\right) \ .$$

In particular, $\mathcal{L}_{\dot{x}}^{\{y\}} = 0$ implies $\dot{x} = -\sum_{i=1}^N m_i\dot{y}_i$. Since y_i are periodic by definition, then $T^{-1}\int_0^T \dot{x}dt = \overline{v}(0) = 0$. □

We now generalize the energy dissipation in the linear case (2.7) for general $\mathbf{L}_\beta$ mechanical systems. We shall denote by $\overline{D}$ the minimal action, and refer to it as the *energy dissipated along a cycle*:

$$\overline{D} = \min_{x=x(t)} \frac{1}{T}\int_0^T \mathcal{L}^{\{y\}}(\dot{x},t)dt = -\frac{1}{T}\int_0^T \mathcal{H}^{\{y\}}(0,t)dt \ . \tag{2.9}$$

Next, we define the efficiency of a mobilized system $\boldsymbol{y} := y_1, \ldots y_N$. This should indicate the ratio of the distance transversed per cycle to the dissipated energy. The right scaling for this turns out to be

$$e_L(y_1,\ldots y_N) := \frac{L\left(\overline{v}(0)\right)}{\overline{D}} = -\frac{L\left(\frac{1}{T}\int_0^T \mathcal{H}_p^{\{y\}}(0,t)dt\right)}{T^{-1}\int_0^T \mathcal{H}^{\{y\}}(0,t)dt} \ . \tag{2.10}$$

In fact, it can be proven that

Lemma 2.1. *If L is a convex function, then for any mechanical system composed of N periodic orbits $y_i(t) = y_i(T+t), i = 1, \ldots N$,*

$$0 < e_L(y_1,\ldots y_N) \leq 1 \ .$$

If, moreover, L is strictly convex, then $e_L < 1$.

Proof. Let $x(t)$ be the *any* orbit. Then,

$$T^{-1}\int_0^T \mathcal{L}^{\{y\}}(\dot{x},t)dt = T^{-1}\sum_{i=1}^N m_i \int_0^T L(\dot{y}_i(t)+\dot{x}(t))dt \ .$$

By Jensen's inequality, the normalization condition $\sum m_i = 1$, the periodicity of y_i and the convexity of L:

$$T^{-1}\sum_{i=1}^N m_i \int_0^T L(\dot{y}_i(t)+\dot{x}(t)) \geq L\left(T^{-1}\int_0^T \sum_i^N m_i(\dot{y}_i+\dot{x})dt\right) = L\left(T^{-1}\int_0^T \dot{x}dt\right) \ , \tag{2.11}$$

so $0 < e_L \leq 1$. Now, if L is strictly convex then the equality in (2.11) holds if and only if $N=1$ or $y_i \equiv y_j$ for all $1 \leq i,j \leq N$. But, in the later cases, the optimal orbit x is evidently equal to any component y_i, so it is a periodic function and $\overline{v}(0)=0$. Hence the system is not mobilized and efficiency is not defined (or $e_L = 0$ altogether). □

Assume now that a convex Lagrangian L is given, as well as $N \in \mathbb{N}$ and $\{m_1, \ldots m_N\} \in \mathbb{R}^{+,N}$, $\sum_1^N m_i = 1$. Let

$$\Lambda_L := \left\{\boldsymbol{y} = (y_1, \ldots y_N) \in C^1([0,1];\mathbb{R}^N) \ ; \boldsymbol{y}(0) = \boldsymbol{y}(1) \ .\right\} \tag{2.12}$$

Let

$$\overline{e}_L := \sup_{\boldsymbol{y}\in\Lambda} e_L(\boldsymbol{y}) \ . \tag{2.13}$$

We know that $\overline{e}_L \leq 1$. The intriguing questions are

i) Is $\overline{e}_L = 1$?

ii) If $\overline{e}_L < 1$, can the supremum in (2.13) be achieved in some sense?

We try to answer these questions in the special case of homogeneous Lagrangians.

3 Homogeneous Lagrangians

To fix the idea, let us concentrate on the case $L(s) = |s|^\sigma$ for some $\sigma > 1$. We shall further assume a unit period $T = 1$.

Let

$$H^\sigma := \left\{x = x(t) : [0,1] \to \mathbb{R} \ , \ 0 \leq t \leq 1 \ ; \ \int_0^1 |\dot{x}|^\sigma < \infty\right\} \ . \tag{3.1}$$

Let also

$$\Lambda_\sigma := \left\{\boldsymbol{y} \in (H^\sigma)^N \ \ , \boldsymbol{y}(0) = \boldsymbol{y}(1)\right\} \ . \tag{3.2}$$

Given $m_i = m \in (0,1)$, $\sum_1^N m_i = 1$ and $\boldsymbol{y} = (y_1, \ldots y_N) \in \Lambda_\sigma$ consider the Lagrangian

$$\mathcal{L}(\dot{x}, \dot{\boldsymbol{y}}) := \sum_{i=1}^N m_i |\dot{y}_i + \dot{x}|^\sigma \ . \tag{3.3}$$

Define also for any such $\boldsymbol{y}$ and $p \in \mathbb{R}^N$,

$$\mathcal{H}(p, \dot{\boldsymbol{y}}) := \sup_{\xi \in \mathbb{R}^N} \{p \cdot \xi - \mathcal{L}(\xi, \dot{\boldsymbol{y}})\} \ , \tag{3.4}$$

and

$$\mathbf{H}^{\boldsymbol{y}}(p) := \int_0^1 \mathcal{H}(p, \dot{\boldsymbol{y}}(t)) dt \tag{3.5}$$

Also, define:

$$u(\dot{\boldsymbol{y}}) := \left. \frac{\partial \mathcal{H}(p, \dot{\boldsymbol{y}})}{\partial p} \right|_{p=0} \tag{3.6}$$

and

$$< u >_{\boldsymbol{y}} := \int_0^1 u(\dot{\boldsymbol{y}}(t)) dt \ . \tag{3.7}$$

We obtain via (3.4-3.7), (2.10) and (2.13)

$$e_\sigma(\boldsymbol{y}) := -\frac{|< u >_{\boldsymbol{y}}|^\sigma}{\mathbf{H}^{\boldsymbol{y}}(0)} \quad ; \quad \overline{e}_\sigma = \sup_{\boldsymbol{y} \in \Lambda_\sigma} e_\sigma(\boldsymbol{y}) \ . \tag{3.8}$$

One may wonder if, under the homogeneity condition, the mechanical system is mobilized, i.e $\overline{e}_\sigma > 0$. The first result we claim is that this condition is equivalent to a property of the function u as defined in (3.6), namely

Lemma 3.1. *The homogeneous system is mobilized if and only if u is not a linear function.*

Proof. If u is a linear function, then for any $\boldsymbol{y} \in \Lambda_\sigma$, $\int u\,(\dot{\boldsymbol{y}})\, dt = 0$ by definition of Λ_σ. Indeed, the condition $\boldsymbol{y}(0) = \boldsymbol{y}(1)$ is equivalent to $\int_0^1 \dot{\boldsymbol{y}}(t) dt = 0$.

Conversely, if u is not a linear function, then there exists $\boldsymbol{y} \in \Lambda_\sigma$ for which $\int_0^1 u\,(\dot{\boldsymbol{y}})\, dt \neq 0$. This implies $e_\sigma(\boldsymbol{y}) > 0$ and, in particular, the system is mobilized. □

The condition of non-linearity of u is rather delicate. In fact,

Lemma 3.2. *If either $\sigma = 2$ or $N = 2$ then u is a linear function.*

Proof. We already know that $\sigma = 2$ (quadratic Lagrangian) is not mobilized for any $N \in \mathbb{N}$ by Theorem 1.

By the homogeneity of L and the definition of u we observe that u satisfies the pair of symmetries:

$$\forall \zeta \in \mathbb{R}^N \ , \ \boldsymbol{e} = (1, \dots 1) \ , \ \textit{and} \ \lambda \in \mathbb{R} \quad , \quad u(\boldsymbol{\zeta} + \lambda \boldsymbol{e}) = u(\boldsymbol{\zeta}) + \lambda$$

$$\forall \alpha \in \mathbb{R}^+ \ , \ \ u(\alpha \boldsymbol{\zeta}) = \alpha u(\boldsymbol{\zeta}) \tag{3.9}$$

We conclude, therefore, that $u(y_1, y_2) = f(y_1 - y_2) + y_1$ holds for some function f of a single variable. From the second equality of (3.9) it follows that

$$f(\alpha(y_1 - y_2)) + \alpha y_1 = \alpha f(y_1 - y_2) + \alpha y_1 \Longrightarrow \alpha f(\zeta) = f(\alpha \zeta)$$

for any $\alpha \in \mathbb{R}$ and $\zeta \in \mathbb{R}$. Hence, f is linear and so is u. □

The main result of this paper is:

3.1 Main result

Theorem 2. *If the homogeneous mechanical system (3.3) is mobilized then there exists a maximizer of e_σ in the set Λ_σ, and $\overline{e}_\sigma < 1$.*

Remark: Lemma 3.2 implies that $N > 2$ and $\sigma \neq 2$ are necessary for the condition of Theorem 2. We conjecture that these conditions are also sufficient. In any case, it is not difficult to construct examples for mobilized homogeneous systems. For example, take $\sigma = 3$, $N = 3$ and $m_1 = m_2 = m_3 = 1/3$. The function $u = u(y_1, y_2, y_3)$ can be readily calculated as the root of a quadratic equation whose coefficients are linear functions of y_i. The discriminant, however, is *not* a complete square, so u is not linear.

In the rest of this section we prove Theorem 2.

From (3.9) we obtain that $\mathcal{H}(0, \dot{\boldsymbol{y}})$ is $\sigma-$homogeneous, that is

$$\alpha \in \mathbb{R} \ , \quad \mathcal{H}(0, \alpha\dot{\boldsymbol{y}}) = |\alpha|^\sigma \mathcal{H}(0, \dot{\boldsymbol{y}})$$

as well as

$$< u >_{\alpha \boldsymbol{y}} = \alpha < u >_{\boldsymbol{y}} \ .$$

In particular

$$e_\sigma(\alpha \boldsymbol{y}) = e_\sigma(\boldsymbol{y}) \ , \ \forall \alpha \in \mathbb{R}, \boldsymbol{y} \in \Lambda_\sigma \ . \tag{3.10}$$

In addition, $e_\sigma(\boldsymbol{y})$ is clearly invariant under diagonal shifts $\boldsymbol{y} \rightarrow \boldsymbol{y}(t) + \beta(t)\boldsymbol{e}$ where $\boldsymbol{e} = (1, \ldots 1) \in \mathbb{R}^N$. Define now

$$\Lambda^0_\sigma := \{\boldsymbol{y} = (y_1, \ldots y_N) \in \Lambda_\sigma \ ; \quad y_1 \equiv 0 \ .\}$$

and

$$S_\sigma = \left\{ \boldsymbol{y} \in \Lambda^0_\sigma \ ; \quad \int_0^1 |\dot{\boldsymbol{y}}(t))|^\sigma \, dt = 1 \right\} \quad ; \quad B_\sigma = \left\{ \boldsymbol{y} \in \Lambda^0_\sigma \ ; \quad \int_0^1 |(\dot{\boldsymbol{y}}(t))|^\sigma \, dt \leq 1 \right\} . \tag{3.11}$$

By the scaling (3.10) and the diagonal shift invariance we conclude that

$$\overline{e}_\sigma := \sup_{\boldsymbol{y} \in \Lambda_\sigma} e_\sigma(\boldsymbol{y}) = \sup_{\boldsymbol{y} \in S_\sigma} e_\sigma(\boldsymbol{y}) = \sup_{\boldsymbol{y} \in B_\sigma - \{0\}} e_\sigma(\boldsymbol{y}) \ . \tag{3.12}$$

Let now $\boldsymbol{y}_j$ be a maximizing sequence of e_σ in S_σ. There is a weak limit $\boldsymbol{y}_\infty \in B_\sigma$ of this sequence. The inequality

$$\lim_{j \rightarrow \infty} \mathbf{H}^{\boldsymbol{y}_j}(0) \leq \mathbf{H}^{\boldsymbol{y}_\infty}(0)$$

holds since $\mathbf{H}^{\boldsymbol{y}}(0)$ is upper-semi-continuous, but we do not have the same claim for $< u >_{\boldsymbol{y}}$. So, we cannot prove that $vecy_\infty$ is a maximizer of e_σ.

Another problem is that we may have $\boldsymbol{y}_\infty = \mathbf{0}$, so $e_\sigma(\boldsymbol{y}_\infty)$ is not defined at all. As an example, let $\boldsymbol{y} \in \Lambda_\sigma$ and assume that $< u >_{\boldsymbol{y}} \neq 0$. This, in particular, implies that $x(t) := \int^t u(\dot{\boldsymbol{y}}) = \tilde{x}(t) + \lambda t$ where $\tilde{x}$ is a periodic function and $\lambda \neq 0$. If we replace $\boldsymbol{y}$ by $\boldsymbol{y}_j = \boldsymbol{y}_j(t) := j^{-1}\boldsymbol{y}(jt)$ for $j \in \mathbb{N}$, using the periodicity of $\boldsymbol{y}$ to define $\boldsymbol{y}_j$ on $(0, 1)$, the following claims are straightforward:

a) $\boldsymbol{y}_j \in \Lambda_\sigma$ for any $j \in \mathbb{N}$.

b) $\lim_{j\to\infty} \boldsymbol{y}_j = 0$ weakly.

c) $x_j = j^{-1}x(jt) = j^{-1}\tilde{x}(jt) + \lambda t = \int^t u\left(\dot{\boldsymbol{y}}_j\right)$.

d) $< u >_{\boldsymbol{y}_j} = \lambda$ for any $j \in \mathbb{N}$.

In particular, we find out that $e_\sigma(\boldsymbol{y}_j) = e_\sigma(\boldsymbol{y})$, while e_σ is not defined for the weak limit $\lim_{j\to\infty} \boldsymbol{y}_j = \mathbf{0}$.

3.2 Relaxation

To overcome the last difficulty we shall extend the definition (3.5-3.7) as follows: Let

$$\mathcal{P}^N := \{\textit{Probability Borel measures on } \mathbb{R}^N\}$$

Given $\nu \in \mathcal{P}^N$, let

$$\mathbf{H}^\nu(p) := \int_{\mathbb{R}^N} \mathcal{H}(p, v)\nu(dv) \tag{3.13}$$

and

$$< u >_\nu := \int_{\mathbb{R}^N} u(v)\nu(dv) \ . \tag{3.14}$$

The generalization of (3.8) is given by

$$e_\sigma(\nu) := -\frac{|< u >_\nu|^\sigma}{\mathbf{H}^\nu(0)} = \frac{|< u >_\nu|^\sigma}{\int_{\mathbb{R}^n} \sum_1^N m_i |v_i + u(\boldsymbol{v})|^\sigma \nu(d\boldsymbol{v})} \tag{3.15}$$

We shall further extend the definitions of Λ_σ and Λ_σ^0 as follows:

$$\overline{\Lambda}_\sigma := \left\{ \nu = \nu(dv) \in \mathcal{P}^N \ \ ; \ \int_{\mathbb{R}^N} |v|^\sigma \nu(dv) < \infty \ \ ; \ \ \int_{\mathbb{R}^N} v_i \nu(dv) = 0 \ , 1 \leq i \leq N \right\} \tag{3.16}$$

$$\overline{\Lambda}_\sigma^0 := \left\{ \nu \in \overline{\Lambda}_\sigma \ ; \ \nu = \delta_{v_1}\mu(dv_2, \ldots dv_N) \ \ \textit{where } \mu \in \mathcal{P}^{N-1} \right\} \tag{3.17}$$

Lemma 3.3. *For any $\boldsymbol{y} \in \Lambda_\sigma$ (res. $\boldsymbol{y} \in \Lambda_\sigma^0$) there exists $\nu \in \overline{\Lambda}_\sigma$ (res. $\nu \in \overline{\Lambda}_\sigma^0$) so that $e_\sigma(\nu) = e_\sigma(\boldsymbol{y})$. Conversely, for any $\nu \in \overline{\Lambda}_\sigma$ (res. $\nu \in \overline{\Lambda}_\sigma^0$) there exists $\boldsymbol{y} \in \Lambda_\sigma$ (res. $\boldsymbol{y} \in \Lambda_\sigma^0$) so that $e_\sigma(\nu) = e_\sigma(\boldsymbol{y})$.*

Remark: The measure ν associated with $\boldsymbol{y}$ is related to *Young measure*. In general, however, Young measures are used to study the oscillatory behavior of a weak limit of $\mathbb{L}^\infty$ sequences (see, e.g., [E]).

Proof. For the first part, define $\nu(dv) = \int_0^1 \prod_1^N \delta_{v_i - \dot{y}_i(t)}$. To elaborate, the measure ν corresponding to $\boldsymbol{y}$ is obtained by its application on test functions $\phi \in C_0\left(\mathbb{R}^N\right)$:

$$\int_{\mathbb{R}^N} \phi(v)\nu(dv) = \int_0^1 \phi\left(\dot{\boldsymbol{y}}(t)\right) dt \ .$$

If $\boldsymbol{y} \in \Lambda_\sigma$ then the above equality also extend to $\phi(\dot{\boldsymbol{y}}) = \dot{\boldsymbol{y}}$ and $\phi(\dot{\boldsymbol{y}}) = |\dot{\boldsymbol{y}}|^\sigma$. In particular

$$\int_{\mathbb{R}^N} v\nu(dv) = \int_0^1 \dot{\boldsymbol{y}}(t)dt = 0 \ ; \quad \int_{\mathbb{R}^N} |v|^\sigma \nu(dv) = \int_0^1 |\dot{\boldsymbol{y}}|^\sigma (t)dt < \infty \ .$$

Finally, the equalities

$$\mathbf{H}^{\boldsymbol{y}}(0) = \mathbf{H}^\nu(0) \quad ; \quad < u >_\nu = < u >_{\boldsymbol{y}} \tag{3.18}$$

hold under this identification.

For the second part we use Theorem 2.1 of [Am] to observe the following:
For any such ν there exists a Borel function $T : [0,1] \to \mathbb{R}^N$ which push forward the Lebesgue measure dt on $[0,1]$ to ν. That is, for any test function $\phi \in C_0\left(\mathbb{R}^N\right)$,

$$\int_0^1 \phi(T(t))dt = \int_{\mathbb{R}^N} \phi(v)\nu(dv) \ .$$

In fact, Theorem 2.1 of [Am] claims the equality between the infimum of Monge and the minimum of the Kantorowich transport plan of probability measures μ_1 to μ_2 on $\mathbb{R}^N$, provided μ_1 has no atoms. This implies, in particular, that the set of Borel mappings transporting μ_1 to μ_2 is not empty. In our case we use this result under the identification of μ_1 with the Lebesgue measure on $[0,1]$ (considered as a Hausdorff measure in $\mathbb{R}^N$ for the embedded interval) and μ_2 with ν.

So, set $\boldsymbol{y}(t) := \int^t T(s)ds$. Is is absolutely continues function satisfying $\dot{\boldsymbol{y}} = T$ a.e. The definition (3.16) implies also that $\int_0^1 |\dot{\boldsymbol{y}}|^\sigma = \int |v|^\sigma \nu(dv) < \infty$, as well as $\int_0^1 \dot{\boldsymbol{y}} dt = \int v\nu(dv) = 0$, which yields the periodicity of $\boldsymbol{y}$. The equality (3.18) holds under this identification as well. □

3.3 Proof of Theorem 2

Define now

$$\psi(\boldsymbol{v}) := |u(\boldsymbol{v})| + \sum_{i=2}^N |v_i + u(\boldsymbol{v})| \ , \ \boldsymbol{v} = (0, v_2, \dots v_N) \ . \tag{3.19}$$

We shall summertime some properties of ψ which will be needed later:

Lemma 3.4. *There exists $A > 0$ so that*

$$A^{-1}|\boldsymbol{v}| \leq |\psi(v)| \leq A|\boldsymbol{v}| \quad \forall \boldsymbol{v} = (0, v_2, \dots v_N) \ . \tag{3.20}$$

In addition, for any $\nu \in \overline{\Lambda}^0_\sigma$,

$$A^{-1} \int_{\mathbb{R}^N} |\psi(v)|^\sigma \nu(dv) < -\mathbf{H}^\nu(0) \leq \ A \int_{\mathbb{R}^N} |\psi(v)|^\sigma \nu(dv). \tag{3.21}$$

Proof. The estimate (3.20) follows from the homogeneity of u, namely $u(\alpha\boldsymbol{v}) = \alpha u(\boldsymbol{v})$, as well as from the evident property $u(0, v, \dots v) = 0 \Longleftrightarrow v = 0$. The estimate (3.21) follows from (3.3, 3.4) and (3.13), as well as (3.20). □

Given $1 < q < \sigma$, let $\overline{S}_\sigma^q$ be the unit sphere (res. $\overline{B}_\sigma^q$ the unit ball) defined by

$$\overline{S}_\sigma^q = \left\{ \nu \in \overline{\Lambda}_\sigma^0 \ ; \ \int_{\mathbb{R}^{N-1}} |\boldsymbol{v}|^q \nu(dv) = 1 \right\} \quad ; \quad \overline{B}_\sigma^q = \left\{ \nu \in \overline{\Lambda}_\sigma^0 \ ; \ \int_{\mathbb{R}^{N-1}} |\boldsymbol{v}|^q \nu(dv) \le 1 \right\} \tag{3.22}$$

The analogous of (3.12) also holds due to the scaling invariance (3.10):

Lemma 3.5. $\overline{e}_\sigma = \sup_{\nu \in \overline{S}_\sigma^q} e_\sigma(\nu)$.

Proof. We only have to show that any $\nu \in \overline{\Lambda}_\sigma$ (res. $\nu \in \overline{\Lambda}_\sigma^0$) can be transformed into $\hat{\nu} \in \overline{S}_\sigma$ (res. $\hat{\nu} \in \overline{S}_\sigma^0$) so that $e_\sigma(\nu) = e_\sigma(\hat{\nu})$. Let $\hat{\nu}(dv) = \beta^N \nu(\beta dv)$ for $\beta > 0$. The homogeneity properties (3.10) imply that, indeed, $e_\sigma(\hat{\nu}) = e_\sigma(\nu)$ for any such β. In addition, $\int v\hat{\nu}(dv) = \beta^{-1} \int v\nu(dv) = 0$ if $\nu \in \overline{\Lambda}_\sigma$. However, $\int |v|^q \hat{\nu}(dv) = \beta^{-q} \int |v|^q \nu(dv)$, so $\hat{\nu} \in \overline{S}_\sigma$ if $\beta = \left(\int |v|^q \nu(dv) \right)^{1/q}$. □

Let ν_j be a maximizing sequence of e_σ in $\overline{S}_\sigma^0$. Since the $q > 1$ moments of ν_j are uniformly bounded, this sequence is compact (tight) in the weak topology of measures. Let ν_∞ be the weak limit of ν_j. Since $|u(v)| \le A|v|$ for some $A > 0$ it follows that the sequence of (signed) measures $u(v)\nu_j(dv)$ is tight as well, and that

$$\lim_{j\to\infty} u(v)\nu_j(dv) = u(v)\nu_\infty(dv) \ . \tag{3.23}$$

On the other hand, it is not a-priori evident that $\nu_\infty \in \overline{S}_\sigma^0$, since the sequence $|v|^q \nu_j$ is not necessarily tight, so we only know

$$\int |v|^q \nu_\infty(dv) \le 1 \ . \tag{3.24}$$

We claim, however, that, in fact, the σ moments of ν_j are uniformly bounded. For, if $\int |v|^\sigma \nu_j(dv) \to \infty$, then by (3.20, 3.21), also $-\mathbf{H}^{\nu_j}(0) \to \infty$. Since we also now that $< u >_{\nu_j}$ are uniformly bounded, then

$$e_\sigma(\nu_j) = -\frac{< u >_{\nu_j}}{\mathbf{H}^{\nu_j}(0)} \to 0 \ ,$$

contradicting the assumption that ν_j is a maximum sequence for e_σ.

Since $q < \sigma$ by assumption it follows that $|v|^q \nu_j$ is a tight sequence as well, so there is, in fact, an equality in (3.24). In particular, it follows that $\nu_\infty \neq \delta_0$. Moreover, using (3.20, 3.21) for ν_∞ and the equality in (3.24) again, we also have

$$-\mathbf{H}^{\nu_\infty}(0) > 0 \ .$$

On the other hand, $-\mathbf{H}^\nu(0)$ is nothing but the expectation of ν with respect to a positive, continuous function $\mathcal{L}(v) = \sum_{i=2}^N m_i |v_i - u(v)|^\sigma$. Hence

$$-\lim_{j\to\infty} \mathbf{H}^{\nu_j}(0) := \lim_{j\to\infty} \int \mathcal{L}(v)\nu_j(dv) \ge \int \mathcal{L}(v)\nu_\infty(dv) := -\mathbf{H}^{\nu_\infty}(0) \ .$$

This, together with (3.23), implies that

$$e_\sigma(\nu_\infty) \geq \lim_{j\to\infty} e_\sigma(\nu_j)\ . \tag{3.25}$$

Again, ν_j is a maximizing sequence for e_σ hence there is an equality in (3.25), so $\overline{e}_\sigma = e_\sigma(\nu_\infty)$. By the second part of Lemma 3.3 we obtain the existence of $\boldsymbol{y}_\infty \in \Lambda^0_\sigma$ for which $e_\sigma(\nu_\infty) = e_\sigma(\boldsymbol{y}_\infty)$. $\square$

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